TWO-DIMENSIONAL NONSTATIONARY SOLUTIONS OF THE EQUATIONS OF MAGNETOHYDRODYNAMICS

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UDC 533.95:538.4

We consider a simple two-dimensional plane solution of the equations of magnetohydrodynamics given earlier in general form by A. G. Kulikovskii [1]. The plane cumulation of material near a magnetic neutral line is a special case of this solution and is of interest for various fast processes in a plasma such as the z-pinch and solar flares [2].

There has recently been an increased interest in two-dimensional magnetohydrodynamic motions of matter in connection with a number of intensively studied physical and astrophysical phenomena, for example, the plasma focus in a z-pinch and solar chromospheric flares. It is significant also that as a consequence of the progress of computational mathematics all these phenomena have been studied largely by numerical methods. Thus it is useful to examine those few analytic solutions of two-dimensional magnetohydrodynamics which have even an indirect relation to actual magnetohydrodynamics problems in a complete physical formulation. We examine a certain class of simple solutions of the equations of magnetohydrodynamics for two-dimensional plane nonstationary motion. Two subclasses of solutions with completely different physical meanings can be distinguished. The subclass including solutions with a cumulative plane compression of material near a magnetic neutral line is of particular interest.

1. Let us consider the two-dimensional plane nonstationary problem of magnetohydrodynamics. Suppose the motion of the material occurs in the x,y plane, i.e., the z component of the velocity $v_Z = 0$. If in addition there is no z component of the magnetic field, $B_Z = 0$, and all quantities depend on the coordinates x and y and the time t, then only the z components of the electric field E_Z and the current density j_Z can be different from zero. Suppose further that Ohm's law has its simplest form

$$\mathbf{j} = \sigma(\mathbf{E} + c^{-1}[\mathbf{v}\cdot\mathbf{B}]). \tag{1.1}$$

In this formulation of the problem the electromagnetic field can be described by a single component of the vector potential A, $A_Z = A$:

$$B_x = \frac{\partial A}{\partial y}, \quad B_y = -\frac{\partial A}{\partial x}, \quad E_z = -\frac{1}{c}\frac{\partial A}{\partial t}. \tag{1.2}$$

The scalar potential φ is set equal to zero as a consequence of the gauge invariance of the electromagnetic field [3]. Then if the conductivity of the material σ does not depend on the spatial coordinates, the magnetohydrodynamic equations can be written in the form

$$\frac{dA}{dt} = \frac{c^2}{4\pi\sigma}\Delta A, \qquad \rho\frac{d\mathbf{v}}{dt} = -\nabla p - \frac{1}{4\pi}\Delta A\nabla A$$
$$\frac{d\rho}{dt} + \rho \operatorname{div}\mathbf{v} = 0 \tag{1.3}$$

where the functions A, v, and ρ depend on x, y, and t. The equation for the component of the vector potential A in (1.3) is identical with the corresponding equation in [2] if (1.1) is taken into account.

2. The system of equations (1.3) has a certain class of simple solutions noted by A. G. Kulikovskii [1]. We assume that the pressure p is a function only of the density ρ. The conductivity of the material σ, which in general depends on the pressure and density, then becomes a function only of the density. Of course this

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 2, pp. 3-7, March-April, 1972. Original article submitted October 7, 1971.

assumption is compatible with Eqs. (1.3) only if the density is independent of the spatial coordinates. We consider the class of solutions

$$A = \mathbf{r}a_+\mathbf{r} + \alpha, \quad \mathbf{v} = U\mathbf{r}, \quad \rho = \rho(t), \quad p = p(\rho) \tag{2.1}$$

where $\mathbf{r} = (x, y)$ is the radius vector, and the matrices $a_+ = \| a_{ij} \|$ and $U = \| u_{ij} \|$ (i, j = 1, 2). We note that $a_{12} = a_{21}$. The time dependence of the nine functions $a_{ij}(t)$, $u_{ij}(t)$, $\alpha(t)$, and $\rho(t)$ (i, j = 1, 2) is determined by a system of ordinary differential equations obtained by substituting the expressions from (2.1) into the original system (1.3). In matrix form these equations are

$$\begin{gathered} U' + U^2 = -\frac{Spa_+}{\pi\rho}a_+, \quad \rho' = -SpU\rho \\ a_+' + a_+^{*\prime} + 2(a_+U + U^*a_+^*) = 0, \quad \alpha' = \frac{c^2}{2\pi\sigma}Spa_+ . \end{gathered} \tag{2.2}$$

Here Spa_+ and SpU are respectively the traces of the matrices a_+ and U; an asterisk denotes the transpose of a matrix, and the primes indicate differentiation with respect to the time.

It is clear from Eqs. (2.2) that the case of finite conductivity of the material is only very slightly different from the limiting case of infinite conductivity. When $\sigma < \infty$, the solution contains an additional function $\alpha(t)$ determined from the last equation of (2.2) after solving the rest of the system. In the limiting case when $\sigma \to \infty$, $\alpha = 0$. Thus according to (1.2) only an addition to the z component of the electric field results from taking account of the finite conductivity of the material. The magnetic field and the motion of the material are the same as in the limiting case.

It is expedient to write Eqs. (2.2) in dimensionless form. To do this we note that the initial conditions of the problem, which are obtained from (2.1) for t = 0, contain two dimensional constants: ρ_0, the initial density of the material, and $a_0 = a_{11}$,* the first coefficient in the quadratic form for the function A. It is easy to see that the combination

$$t_0 = (\pi\rho_0)^{1/2} a_0^{-1} \tag{2.3}$$

has the dimensions of time. Then we obtain from (2.2) the dimensionless equations for the dimensionless functions

$$\bar{a}_{ij} = a_0^{-1}a_{ij}, \quad \bar{u}_{ij} = t_0u_{ij}, \quad \bar{\rho} = \rho_0^{-1}\rho . \tag{2.4}$$

We simplify the notation for the dimensionless functions in (2.4) by omitting the bars over the quantities. Unless specifically noted we henceforth consider the dimensionless quantities

$$\begin{gathered} U' + U^2 = -Spa_+a_+\rho^{-1}, \quad \rho' = -SpU\rho \\ a_+' + a_+^{*\prime} + 2(a_+U + U^*a_+^*) = 0, \quad \alpha' = \tau_0 Spa_+ \end{gathered} \tag{2.5}$$

where now primes denote differentiation with respect to the dimensionless time $\tau = t/t_0$. In (2.5) $\tau_0 = t_0/t'_0$, with $t'_0 = l_0^2 2\pi\sigma/c^2$, where l_0 is an arbitrary unit of length entering into the definition of the dimensionless function $\bar{\alpha} = \alpha/l_0^2 a_0$.

3. In spite of the extreme simplicity of the spatial dependence of the solution under consideration it has an interesting physical meaning. Without exhausting all the forms of the solutions of (2.5) (determined by six dimensionless numbers – the initial values of a_{ij} and u_{ij}) we indicate two particular subclasses of solutions. If the matrices U and a_+ in the initial conditions are diagonal ($a_{12}(0) = u_{21}(0) = 0$), then according to (2.5) they are diagonal for all $\tau > 0$:

$$a_{12} = u_{12} = u_{21} = 0 . \tag{3.1}$$

In the present case the moving material does not cross the coordinate axes and the magnetic lines of force are perpendicular to the axes. A simplified system of equations with $\tau_0 = 0$ is obtained from (2.5) by taking account of (3.1):

$$\begin{gathered} \rho(u_{11}' + u_{11}^2) = -a_{11}(a_{11} + a_{22}), \qquad \rho(u_{22}' + u_{22}^2) = -a_{22}(a_{11} + a_{22}), \\ \rho' + \rho(u_{11} + u_{22}) = 0 \\ a_{11}' + 2a_{11}u_{11} = 0, \qquad a_{22}' + 2a_{22}u_{22} = 0 . \end{gathered} \tag{3.2}$$

* Dimensional considerations [4] and the fact that the initial conditions contain only the two indicated dimensional constants lead to the construction of the class of solutions written in (2.1).

If the signs of a_{11} and a_{22} are opposite initially, the system of equations (3.2) describes the nonstationary motion of material near a magnetic neutral line. The most important property of the motion is the cumulative compression of material occurring during the finite time $t_c \sim t_0$. We show the cumulative character of the general solution of Eqs. (3.2). To do this we introduce new functions ξ and η:

$$a_{11} = \xi^{-2}, \qquad a_{22} = -\eta^{-2}. \tag{3.3}$$

We denote the initial values of ξ and η by ξ_0 and η_0. Without loss of generality we can assume that $\xi, \eta > 0$. Then from the last three Eqs. of (3.2)

$$\rho = (\xi\eta)^{-1}\xi_0\eta_0, \quad u_{11} = \xi'\xi^{-1}, \quad u_{22} = \eta'\eta^{-1}. \tag{3.4}$$

The remaining two equations of system (3.2) give two second-order differential equations for ξ and η:

$$\xi'' = -\frac{\eta}{\xi_0\eta_0}(\xi^{-2} - \eta^{-2}), \qquad \eta'' = \frac{\xi}{\xi_0\eta_0}(\xi^{-2} - \eta^{-2}). \tag{3.5}$$

For $\xi_0 > \eta_0$ and $\xi'_0 > \eta'_0$ it follows from Eqs. (3.5) that $\xi''_0 > 0$ and $\eta''_0 < 0$. This clearly indicates a further strengthening of the inequality $\xi > \eta$. From (3.5) the second derivatives ξ'' and η'' do not change sign right up to the singular point $\tau = \tau_c$ where $\eta(\tau_c) = 0$. If $\xi_0 < \eta_0$ and $\xi'_0 < \eta'_0$, on the other hand, the inequality $\xi < \eta$ is strengthened and a singularity occurs where ξ vanishes. All other types of initial conditions ($\xi_0 > \eta_0$, $\xi'_0 < \eta'_0$ or $\xi_0 < \eta_0$, $\xi'_0 > \eta'_0$) lead to one singularity or the other depending on whether or not the inequality of the first derivatives changes before the ξ and η curves intersect.* We note that the time of the singularity τ_c is always finite since the corresponding second derivative is strictly negative. The cumulation corresponding to the singularity of the solution demonstrated above is characterized by the unbounded increase of ρ, a_{11}, and u_{11} (as $\xi \to 0$) or of ρ, a_{22}, and u_{22} (as $\eta \to 0$). The cumulative compression occurs perpendicular to the y, z or x, z planes. The numerical integration of Eqs. (3.2) and the investigation of the physical properties of the solutions obtained are given in [2].

The other subclass of solutions with completely different physical properties is obtained by assuming that in the initial conditions the diagonal elements of the matrix U are zero and that matrix a_+ is diagonal as before. Then from (2.5) there follow first that the initial structure of the matrices is preserved, and second that the remaining components are uniquely determined

$$\begin{aligned} a_{12} &= u_{11} = u_{22} = 0, & a_{11} &= a_{22} = a \\ u_{12} &= -u_{21} = u, & u &= \pm\sqrt{2}a. \end{aligned} \tag{3.6}$$

The physical meaning of the stationary solution (3.6) is elementary. Material rotates about the z axis with a certain constant angular velocity, and the magnetic lines of force are concentric circles. The electric current density corresponds to the angular velocity so that at all points of space the centrifugal and ponderomotive forces compensate one another:

$$\frac{1}{c}[\mathbf{j}\cdot\mathbf{B}] + \rho\frac{d\mathbf{v}}{dt} = 0. \tag{3.7}$$

The identity (3.7) is easy to prove by taking account of the fact that $4\pi\,\mathbf{j} = c$ curl $\mathbf{B}$ and using Eqs. (1.2), (2.1), (2.3), (2.4), and (3.2).

4. The solution discussed is useful in investigating the plane cumulation of material near a magnetic neutral line. In a specific magnetohydrodynamics boundary-value problem this solution can, under certain restrictions, be used as an expansion near the neutral line. Such an expansion is particularly important when there is a cumulative singularity of the general solution obtained by numerical methods.

The nature of the cumulative singularity is very important, particularly for the physical interpretation of the solution. It is interesting that it is relatively simple to establish the leading terms in the time dependence of all quantities close to the cumulation time τ_c. Substitution into the complete equations (2.5) shows

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* The special case $\xi_0 = \eta_0$ is not an exception since in satisfying the inequality $\xi'_0 > \eta'_0$ at times near $\tau = 0$ the inequalities $\xi'' > 0$ and $\eta'' < 0$ will hold. All the rest of the argument can be repeated word for word. Actually the solution with the initial conditions $\xi_0 = \eta_0$ and $\xi'_0 = \eta'_0$ turns out to be a separate case. This solution does not have singularities and corresponds to a uniform expansion or compression of material without the participation of ponderomotive forces ($j_z = 0$).

that the particular solution

$$a_+ = \begin{Vmatrix} \frac{1}{3}\frac{\beta}{\gamma}\Delta^{-4/3}, & \frac{2\beta}{\sqrt{3\gamma}}\Delta^{-1/3} \\ \frac{2\beta}{\sqrt{3\gamma}}\Delta^{-1/3}, & \beta\Delta^{2/3} \end{Vmatrix}, \quad U = \begin{Vmatrix} -\frac{2}{3}\Delta^{-1}, & \sqrt{\frac{\gamma}{3}} \\ \sqrt{\frac{\gamma}{3}}, & \gamma\Delta \end{Vmatrix} \tag{4.1}$$

$$\rho = \frac{1}{2}\frac{\beta^2}{\gamma^2}\Delta^{-2/3}$$

where $\Delta = \tau_c - \tau$ and β and γ are arbitrary constants, satisfies all the equations up to terms which are second order in comparison with the principal terms. For the particular solution (4.1) the dimensional physical quantities of (1.2) and (2.1) behave as follows:

$$\begin{aligned} &v_x \sim \Delta^{-1}x, \quad B_x \sim \Delta^{-1/3}x, \quad B_y \sim \Delta^{-4/3}x \\ &\rho \sim \Delta^{-2/3}, \quad j_z \sim \Delta^{-4/3}, \quad E_z \sim \Delta^{-7/3}x^2 . \end{aligned} \tag{4.2}$$

It should not be assumed in general that the cumulative solution is peculiar to subclass (3.1) and (3.2). In a broader sense it also satisfies the complete system of equations (2.6), as does the particular solution (4.1) presented above, since nondiagonal elements of the matrices a_+ and U appear in (4.1).

In conclusion we note that as a result of certain transformations solution (2.1), which refers to an unbounded physical system, can be applied to a volume of material bounded by a cylindrical surface [5]. It is also possible to go from (4.1) to a similar solution for an incompressible liquid found earlier [6]. In the latter case the spatial dependence of the pressure $p \sim bx^2 + cy^2 + d$ must be taken into account in Eqs. (1.3) and a bounded volume of material with the pressure specified on its surface must be considered.

The author thanks M. A. Leontovich and S. I. Syrovat-skii for valuable discussions and L. V. Ovsyannikov for a critique and a number of helpful comments.

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